

An improved fountain theorem and its application

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Abstract

The main aim of the paper is to prove a fountain theorem without assuming the τ -upper semi-continuity condition on the variational functional. Using this improved fountain theorem, we may deal with more general strongly indefinite elliptic problems with various sign-changing nonlinear terms. As an application, we obtain infinitely many solutions for a semilinear Schrödinger equation with strongly indefinite structure and sign-changing nonlinearity.

MSC 2010: 35J20; 35J60; 35Q55; 58E05

Keywords: Fountain theorem; strongly indefinite functional; elliptic equation; sign-changing potentials; infinitely many solutions

1 Introduction

Since the pioneering works of Bartsch and Willem [2, 3] (see also, [21]), variant fountain theorems are established and which have been used to study the existence of infinitely many solutions for various elliptic problems, see e.g., [2, 3, 4, 5, 6, 12, 22] and the references therein. In order to investigate infinitely many critical points of strongly indefinite functionals, Batkham-Colin [5] established a generalized fountain theorem based on the so-called τ -topology introduced by Kryszewski-Szulkin [13]. For recalling the fountain theorem proved in [5], we introduce some notations and definitions which are also often used in the following sections of the paper.

Let X be a separable Hilbert space and $Y \subset X$ be a closed subspace of X endowed with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let

$$X = Y \bigoplus Z \text{ with } Z = Y^\perp, Y = \overline{\bigoplus_{j=0}^{\infty} \mathbb{R}e_j} \text{ and } Z = \overline{\bigoplus_{j=0}^{\infty} \mathbb{R}f_j}, \quad (1.1)$$

where $\{e_j\}_{j \geq 0}$ and $\{f_j\}_{j \geq 0}$ are orthonormal bases of Y and Z , respectively. Moreover, we define

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$$Y_k := Y \bigoplus \left(\bigoplus_{j=0}^k \mathbb{R} f_j \right) \quad \text{and} \quad Z_k := \overline{\bigoplus_{j=k}^{\infty} \mathbb{R} f_j}, \quad (1.2)$$

and let

$$P : X \rightarrow Y, \quad Q : X \rightarrow Z \quad \text{and} \quad P_k : X \rightarrow Y_{k-1}, \quad Q_k : X \rightarrow Z_k \quad (1.3)$$

be the orthogonal projections. The τ -topology on $X = Y \oplus Z$ introduced in [13] is the topology associated to the following norm

$$\|u\|_{\tau} := \max \left\{ \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle Pu, e_j \rangle|, \|Qu\| \right\}, \quad \text{for } u \in X. \quad (1.4)$$

By the above definition, we see that

$$\|u\|_{\tau} \leq \max \{ \|Pu\|, \|Qu\| \} \leq \|u\| \quad \text{for } u \in X. \quad (1.5)$$

Furthermore, it follows from [13] and the appendix of [17] that, if $\{u_n\} \subset X$ is bounded, then

$$u_n \xrightarrow{n} u \text{ in } \tau\text{-topology} \Leftrightarrow Pu_n \xrightarrow{n} Pu \text{ and } Qu_n \xrightarrow{n} Qu. \quad (1.6)$$

Remark 1.1 *Let*

$$\tilde{e}_j = \begin{cases} f_j, & j = 0, 1, \dots, k-1 \\ e_{j-k}, & j \geq k, \end{cases}$$

and define the following norm

$$\|u\|_{\tau_k} = \max \left\{ \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle P_k u, \tilde{e}_j \rangle|, \|Q_k u\| \right\}, \quad (1.7)$$

then, $\|\cdot\|_{\tau_k}$ and $\|\cdot\|_{\tau}$ are equivalent for all $k \geq 1$. In fact, it is enough to show that $\|\cdot\|_{\tau_1}$ and $\|\cdot\|_{\tau}$ are equivalent. By (1.4),

$$\begin{aligned} \|u\|_{\tau}^2 &\leq \left(\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle u, e_j \rangle| \right)^2 + \sum_{j=0}^{\infty} |\langle u, f_j \rangle|^2 \\ &= \left(\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle u, e_j \rangle| \right)^2 + |\langle u, f_0 \rangle|^2 + \sum_{j=1}^{\infty} |\langle u, f_j \rangle|^2 \\ &\leq 4 \left(\frac{1}{2} |\langle u, f_0 \rangle| + \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} |\langle u, e_{j-1} \rangle| \right)^2 + 4 \sum_{j=1}^{\infty} |\langle u, f_j \rangle|^2 \leq 8 \|u\|_{\tau_1}^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|u\|_{\tau_1} &\leq \frac{1}{2} |\langle u, f_0 \rangle| + \sum_{j=0}^{\infty} \frac{1}{2^{j+2}} |\langle u, e_j \rangle| + \left(\sum_{j=1}^{\infty} |\langle u, f_j \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle u, e_j \rangle| + 2^{\frac{1}{2}} \left(|\langle u, f_0 \rangle|^2 + \sum_{j=1}^{\infty} |\langle u, f_j \rangle|^2 \right)^{\frac{1}{2}} \leq 2^{\frac{3}{2}} \|u\|_{\tau}. \end{aligned}$$

Throughout the paper, for $r_k > 0$ and $\rho_k > 0$, we always set

$$B_k := \{u \in Y_k : \|u\| < \rho_k\} \quad \text{and} \quad N_k := \{u \in Z_k : \|u\| = r_k\}. \quad (1.8)$$

Since X is a Hilbert space, if $\varphi \in C^1(X, \mathbb{R})$, $\nabla\varphi$ is given by the formula

$$\langle \nabla\varphi(u), v \rangle = \varphi'(u)v, \quad \text{for all } v \in X.$$

With the above notations, for an even functional the fountain theorem proved in [5] can be stated as follows

Theorem 1.1 [5, Corollary 13] *Let $\varphi \in C^1(X, \mathbb{R})$ be an even functional satisfying*

- *$\nabla\varphi$ is weakly sequentially continuous, i.e., for any $v \in X$, $\varphi'(u_n)v \xrightarrow{n} \varphi'(u)v$ if $u_n \xrightarrow{n} u$ weakly in X .*
- *φ is τ -upper semi-continuous, i.e., for any $c \in \mathbb{R}$, the set $\varphi_c := \{u \in X : \varphi(u) \geq c\}$ is τ -closed.*
- *For any $c > 0$, φ satisfies $(PS)_c$ condition, i.e., any sequence $\{u_n\} \subset X$ with*

$$\varphi(u_n) \xrightarrow{n} c \quad \text{and} \quad \varphi'(u_n) \xrightarrow{n} 0 \quad \text{in } X' \quad (\text{the dual space of } X)$$

has a convergent subsequence.

Additionally, if there exist $\rho_k > r_k > 0$ such that:

$$(A_1) \quad d_k := \sup_{u \in Y_k, \|u\| \leq \rho_k} \varphi(u) < \infty,$$

$$(A_2) \quad a_k := \sup_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \leq 0,$$

$$(A_3) \quad b_k := \inf_{u \in Z_k, \|u\| = r_k} \varphi(u) \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Then, φ has an unbounded sequence of critical values.

The τ -upper semi-continuity was proposed in [13] for showing a generalized linking theorem. Similar to [13], this condition is also required in the above Theorem 1.1, which is mainly used to construct a suitable vector field. Theorem 1.1 can be used to deal with some strongly indefinite elliptic problems, but the τ -upper semi-continuity assumption requires that the primitive functions of the nonlinearities of the elliptic problems should be positive, see condition (\mathbf{f}_4) in [5]. It is natural to ask what would happen if the nonlinear terms of an elliptic problem change sign and lose the positivity condition (\mathbf{f}_4) ? So, the main aim of the paper is to establish a variant fountain theorem without assuming the τ -upper semi-continuity and then we may answer the above question, see our Theorems 1.2 and 1.3. Our proofs are motivated by the papers [5, 7, 16]. We mention that if the τ -upper semi-continuity of φ is removed, several steps in the proofs for Theorem 1.1 in [5] seem not working any more, for examples:

- We cannot construct the pseudogradient vector by the same way as in [5] since the set $\varphi^{-1}(-\infty, c)$ may not be τ -open now. This difficulty is overcome in this paper by using some ideas from [7].
- To the authors' knowledge, the intersection lemma used in [5] is no longer applicable since the descending flow in our paper has different behavior from that of [5]. In this paper, we use the intersection lemma given in [16] instead.
- We cannot make an explicit mini-max characterization on the critical values of φ because of the lack of τ -upper semi-continuity for φ , then it is hopeless to get infinitely many different critical points of φ by comparing their critical values as that of [5] or [2]. In this paper, we get infinitely many different critical points $\{u_n\}$ of φ by comparing their norm $\|u_n\|$ and proving $\|u_n\| \xrightarrow{n} +\infty$.

Now, we give our improved fountain theorem:

Theorem 1.2 *Let $\varphi \in C^1(X, \mathbb{R})$ be an even functional satisfying $(PS)^c$ condition (i.e., any sequence $\{u_n\} \subset X$ with $\sup_n \varphi(u_n) \leq c$ and $\varphi'(u_n) \xrightarrow{n} 0$ having a convergent subsequence) and let $\nabla \varphi$ be weakly sequentially continuous. For any $k \in \mathbf{N}$, if there exists $\rho_k > r_k > 0$ such that, in addition to the above assumptions (A_1) and (A_3) , there holds*

$$(A_2)' \quad a_k := \sup_{u \in Y_k, \|u\| = \rho_k} \varphi(u) < \inf_{u \in Z_k, \|u\| \leq r_k} \varphi(u),$$

$$(A_4) \quad \sup_{\|u\|_{\tau} < \delta} \varphi(u) \leq C_{\delta} < \infty, \text{ for any } \delta > 0,$$

then, φ has a sequence of critical points $\{u^{k_m}\}$ such that $\lim_{m \rightarrow \infty} \|u^{k_m}\| \rightarrow \infty$.

Remark 1.2 *In our Theorem 1.2, the τ -upper semi-continuity is not assumed. But, we replaced condition (A_2) in Theorem 1.1 by $(A_2)'$ and added a new assumption (A_4) . However, the conditions $(A_2)'$ and (A_4) are easily verified in the applications, see e.g., the proof of our Theorem 1.3.*

With the above Theorem 1.2, we may study the existence of infinitely many solutions for the following Schrödinger equation with strongly indefinite linear part and sign-changing nonlinear term:

$$\begin{cases} -\Delta u + V(x)u = g(x)|u|^{q-2}u + h(x)|u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N), \quad N \geq 3, \end{cases} \quad (1.9)$$

where $1 < q < \frac{p}{p-1} < 2 < p < 2^*$ and $2^* = \frac{2N}{N-2}$, $V(x)$, $g(x)$ and $h(x)$ are functions satisfying

(H_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R}) \cap L^{\infty}$ and 0 lies in a spectrum gap of the operator $-\Delta + V$.

(H_2) $g \in L^{q_0}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ with $q_0 = \frac{2N}{2N-qN+2q}$.

(H₃) $h \in L^{p_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $p_0 = \frac{2N}{2N-pN+2p}$ and $h(x) > 0$ a.e. in \mathbb{R}^N .

Since 0 lies in a gap of the spectrum of $-\Delta + V$, problem (1.9) may be strongly indefinite. The nonlinearity in (1.9) has a super-linear part and a sub-linear part which is usually called concave-convex nonlinearity, some well-known results corresponding to concave-convex nonlinearities can be found in [1, 3] and the references therein. By (H₂), we see that the weight function $g(x)$ may change sign, so, the variational functional of (1.9) does not satisfy the τ -upper semi-continuous assumption.

We mention that there are some papers on the existence of solutions for Schrödinger equations with both sign-changing potential $V(x)$ and indefinite nonlinearities, see, e.g., [8, 9, 10, 14, 19, 20], etc. But the problems discussed in [8, 9, 10, 19, 20] do not have strongly indefinite structure, and in paper [14] the potential $V(x)$ has to be periodic and the weight functions in nonlinear term must satisfy some additional conditions. There seems no results for problem (1.9) under the conditions (H₁)-(H₃). For problem (1.9), we have the following theorem.

Theorem 1.3 *If the conditions (H₁)-(H₃) hold, then problem (1.9) has a sequence of nontrivial solutions $\{u_n\} \subset H^1(\mathbb{R}^N)$ with $\|u_n\|_{H^1} \rightarrow \infty$, as $n \rightarrow \infty$.*

2 Proof of our fountain theorem

In this section, we are going to prove our fountain theorem, that is, Theorem 1.2. For doing this, some lemmas are required.

Lemma 2.1 *Let Y_k and Z_k be defined in (1.2). $\varphi \in C^1(X, \mathbb{R})$ is an even functional and $\nabla\varphi$ is weakly sequentially continuous. If there exist $k \in \mathbb{N}$ and $\rho_k > r_k > 0$ such that conditions (A₁) (A₂)' are satisfied and there holds*

$$b_k := \inf_{u \in Z_k, \|u\|=r_k} \varphi(u) > \sup_{\|u\|_\tau < \delta} \varphi(u), \text{ for some } \delta > 0. \quad (2.1)$$

Then, there exists a sequence $\{u_n^k\} \subset \varphi^{d_k+1} := \{u \in X : \varphi(u) \leq d_k + 1\}$ such that

$$\inf_n \|u_n^k\|_\tau \geq \frac{\delta}{2} \text{ and } \varphi'(u_n^k) \xrightarrow{n} 0 \text{ in } X' \text{ (the dual space of } X\text{)}.$$

In order to prove Lemma 2.1, we need the following deformation lemma:

Lemma 2.2 *Under the assumptions of Lemma 2.1, let*

$$E = \varphi^{d_k+1} \bigcap \{u \in X : \|u\|_\tau \geq \frac{\delta}{2}\}, \quad (2.2)$$

with $\varphi^{d_k+1} := \{u \in X : \varphi(u) \leq d_k + 1\}$ and δ given in (2.1), if there exists $\epsilon \in (0, \frac{1}{2})$ with

$$0 < \epsilon < b_k - \max\{a_k, \sup_{\|u\|_\tau < \delta} \varphi(u)\}, \quad (2.3)$$

such that

$$\|\varphi'(u)\| > \epsilon, \text{ for any } u \in E,$$

then, there exist $T > 0$ and a map $\eta(t, u) \in C([0, T] \times B_k, X)$ with B_k given by (1.8) such that

- (i) $\eta(0, u) = u$ and $\eta(t, -u) = -\eta(t, u)$ for any $u \in B_k$ and $t \in [0, T]$.
- (ii) $\varphi(\eta(t, u))$ is non-increasing in $t \in [0, T]$ for fixed $u \in B_k$.
- (iii) η is τ -continuous (i.e., $\eta(t_m, u_m) \xrightarrow{m} \eta(t, u)$ in τ -topology, if $t_m \xrightarrow{m} t$ and $u_m \xrightarrow{m} u$ in τ -topology) and $\eta(t, \cdot) : B_k \rightarrow \eta(t, B_k)$ is a τ -homeomorphism for any $t \in [0, T]$.
- (iv) $\eta(T, B_k) \subset \varphi^{b_k - \epsilon}$.
- (v) For any $(t, u) \in [0, T] \times B_k$, there exist a neighborhood $W_{(t, u)}$ of (t, u) in the $|\cdot| \times \tau$ -topology such that

$$\{v - \eta(s, v) | (s, v) \in W_{(t, u)} \cap ([0, T] \times B_k)\}$$

is contained in a finite-dimensional subspace of X .

Proof. For $\epsilon > 0$ given in (2.3), let

$$B_R = \{u \in X : \|u\| \leq R\}, \text{ where } R = 2(d_k - b_k + 2\epsilon)/\epsilon + \rho_k + \delta. \quad (2.4)$$

Firstly, we claim that there exists a vector field $\chi : \varphi^{d_k+1} \rightarrow X$ such that

- (a) χ is odd with $\|\chi(u)\| \leq 2/\epsilon$ and $\langle \nabla \varphi(u), \chi(u) \rangle \leq 0$, for any $u \in \varphi^{d_k+\epsilon}$.
- (b) $\chi(u)$ is locally Lipschitz continuous and τ -locally Lipschitz τ -continuous on $\varphi^{d_k+\epsilon}$.
- (c) $\langle \nabla \varphi(u), \chi(u) \rangle < -1$, for any $u \in \varphi^{-1}[b_k - \epsilon, d_k + \epsilon] \cap B_R$.
- (d) For any $u \in \mathcal{W}$, \mathcal{W} is given by (2.10), there exist a τ -open neighborhood $U_u \in \mathcal{N}$ of u such that $\chi(U_u)$ is contained in a finite-dimensional subspace of X .

In fact, by our assumption, $\|\varphi'(u)\| > \epsilon$ for any $u \in E$, we may define

$$\omega(u) = \frac{2\nabla \varphi(u)}{\|\nabla \varphi(u)\|^2}, \text{ for } u \in E \cap B_R,$$

then, there exists a τ -neighborhood $V_u \subset X$ of u such that

$$\langle \nabla \varphi(v), \omega(u) \rangle > 1, \quad \text{for any } v \in V_u \cap B_R. \quad (2.5)$$

Otherwise, if such V_u does not exist, then there exists a sequence $\{v_n\} \subset B_R$ such that $v_n \xrightarrow{\tau} u$ and $\lim_{n \rightarrow \infty} \langle \nabla \varphi(v_n), \omega(u) \rangle \leq 1$. By (1.6) we have $v_n \rightharpoonup u$ weakly in X and this leads to a contradiction since $\nabla \varphi$ is weakly continuous and $\langle \nabla \varphi(u), \omega(u) \rangle = 2$.

Note that B_R is τ -closed [7], thus $X \setminus B_R$ is τ -open, and

$$\mathcal{N} = \{V_u : u \in E \cap B_R\} \cup \{X \setminus B_R\} \quad (2.6)$$

forms a τ -open covering of E .

Since \mathcal{N} is metric, hence paracompact, there exists a local finite τ -open covering $\mathcal{M} = \{M_i : i \in \Lambda\}$, where Λ is an index set, of E finer than \mathcal{N} . If $M_i \subset V_{u_i}$ for some $u_i \in E$, we choose $\omega_i = \omega(u_i)$ and if $M_i \subset X \setminus B_R$, we choose $\omega_i = 0$. Let $\{\lambda_i(u) : i \in \Lambda\}$ be a τ -Lipschitz continuous partition of unity subordinated to \mathcal{M} and let

$$\xi(u) = \sum_{i \in \Lambda} \lambda_i(u) \omega_i, \quad u \in \mathcal{N}.$$

Since the τ -open covering \mathcal{M} of \mathcal{N} is local finite, each $u \in \mathcal{N}$ belongs to finite many sets M_i . Therefore, for every $u \in \mathcal{N}$, the sum $\xi(u)$ is only a finite sum. It follows that, for any $u \in \mathcal{N}$, there exist a τ -open neighborhood $U_u \subset \mathcal{N}$ of u such that $\xi(U_u)$ is contained in a finite-dimensional subspace of X . Then, by the equivalence of norms in a finite-dimensional vector space, we know that there exists $C > 0$ such that

$$\|\xi(v) - \xi(w)\| \leq C \|\xi(v) - \xi(w)\|_\tau, \quad \forall v, w \in U_u. \quad (2.7)$$

On the other hand, by the τ -Lipschitz continuity of λ_i and (1.5), we have that there exists a constant $L_u > 0$ such that

$$\|\xi(v) - \xi(w)\|_\tau \leq L_u \|v - w\|_\tau \leq L_u \|v - w\|, \quad \forall v, w \in U_u. \quad (2.8)$$

Then, from (2.7) and (2.8) we know that $\xi(u)$ is locally Lipschitz continuous and τ -locally Lipschitz τ -continuous. Moreover, by (2.5) and the property of λ_i , we also have that

$$\langle \nabla \varphi(u), \xi(u) \rangle > 1 \text{ and } \|\xi(u)\| < \frac{2}{\epsilon}, \text{ for any } u \in E \cap B_R.$$

Since φ is even, \mathcal{N} is symmetric, we define $\tilde{\xi}(u) := \frac{\xi(u) - \xi(-u)}{2}$ for $u \in \mathcal{N}$, and $\tilde{\xi}(u)$ is odd. For $\delta > 0$ given by (2.1), let $\theta \in C^\infty(\mathbb{R}, [0, 1])$ such that

$$\theta(t) = \begin{cases} 0, & 0 \leq t \leq \frac{2\delta}{3}, \\ 1, & t \geq \delta. \end{cases}$$

Define the vector field $\chi : \mathcal{N} \rightarrow X$ by

$$\chi(u) = \begin{cases} -\theta(\|u\|_\tau) \tilde{\xi}(u), & u \in \mathcal{N}, \\ 0, & \|u\|_\tau \leq \frac{2}{3}\delta. \end{cases} \quad (2.9)$$

It's easy to see that χ is an odd vector field and also well defined on

$$\mathcal{W} := \mathcal{N} \cup \{u \in X : \|u\|_\tau < \delta\}, \quad (2.10)$$

satisfying

$$\|\chi(u)\| \leq \frac{2}{\epsilon} \text{ and } \langle \nabla \varphi(u), \chi(u) \rangle \leq 0, \text{ for any } u \in \mathcal{W}. \quad (2.11)$$

Since $0 < \epsilon < \frac{1}{2}$, we have \mathcal{W} covers $\varphi^{d_k+\epsilon} \cup (X \setminus B_R)$, this shows **(a)**. By the construction of $\chi(u)$, we know that $\chi(u)$ is locally Lipschitz continuous and τ -locally Lipschitz τ -continuous on $\varphi^{d_k+\epsilon}$, and **(b)** is proved.

Moreover, by the choice of ϵ in (2.3), we have

$$\sup_{\|u\|_\tau \leq \delta} \varphi(u) < b_k - \epsilon,$$

i.e.,

$$\{u \in X : \|u\|_\tau \leq \delta\} \subset \varphi^{b_k-\epsilon}.$$

So, by (2.9),

$$\langle \nabla \varphi(u), \chi(u) \rangle < -1, \text{ for any } u \in \varphi^{-1}[b_k - \epsilon, d_k + \epsilon] \cap B_R, \quad (2.12)$$

which implies **(c)**. Then, by the definition of $\chi(u)$, i.e., (2.9), and the properties of $\xi(u)$, we see that **(d)** holds. So, the claim is proved.

Next, we turn to proving **(i)-(v)** of the lemma. For this purpose, we construct a map η through the following Cauchy problem:

$$\begin{cases} \frac{d\eta}{dt} = \chi(\eta) \\ \eta(0, u) = u \in \mathcal{W}. \end{cases} \quad (2.13)$$

By the standard theory of ordinary differential equation in Banach space, we know that the initial problem has a unique solution $\eta(t, u)$ on $[0, \infty)$. Furthermore, the similar argument to the proof of [21, Lemma 6.8] yields that η is τ -continuous. Moreover, $\eta(t, \cdot) : B_k \rightarrow \eta(t, B_k)$ is a τ -homeomorphism for any $t \in [0, T]$. So, part **(iii)** is proved.

Let B_k and B_R be given by (1.8) and (2.4). Taking

$$T = d_k - b_k + 2\epsilon. \quad (2.14)$$

Then, $\{\eta(t, u) : 0 \leq t \leq T, u \in B_k\} \subset B_R$. Indeed, it follows from (2.13) that

$$\eta(t, u) = u + \int_0^t \chi(\eta(s, u)) ds, \text{ for } u \in B_k.$$

By the definition of d_k (see condition (A_1)), we know that $B_k \subset \mathcal{W}$. Then, by (2.3), (2.4) and (2.11), we have for any $u \in B_k$ and $t \in [0, T]$

$$\begin{aligned} \|\eta(t, u)\| &\leq \|u\| + \int_0^t \|\chi(\eta(s, u))\| ds \\ &\leq \|u\| + \int_0^t \frac{2}{\epsilon} ds \leq \|u\| + \frac{2T}{\epsilon} \leq R. \end{aligned}$$

So, **(i)** is obvious by the oddness of $\chi(u)$. By **(a)** we have

$$\frac{d}{dt} \varphi(\eta(t, u)) = \langle \nabla \varphi(u), \chi(u) \rangle \leq 0, \text{ for any } u \in B_k,$$

so, $\varphi(\eta(t, u))$ is non-increasing in $t \in [0, T]$ for fixed $u \in B_k$ and (ii) is proved.

Now, we claim that $\eta(T, B_k) \subset \varphi^{b_k - \epsilon}$. Otherwise, there exists $u \in B_k$ such that

$$\varphi(\eta(T, u)) > b_k - \epsilon. \quad (2.15)$$

Since $\eta(t, u)$ is non-increasing along t , we have

$$\eta(t, u) \in \varphi^{-1}[b_k - \epsilon, d_k + \epsilon] \cap B_R, \text{ for any } t \in [0, T]. \quad (2.16)$$

Then, using (2.12) we see that

$$\begin{aligned} \varphi(\eta(T, u)) &= \varphi(\eta(0, u)) + \int_0^T \langle \varphi'(\eta(s, u)), \chi(\eta(s, u)) \rangle ds \\ &\leq \varphi(\eta(0, u)) + \int_0^T -1 ds \\ &\leq d_k + \epsilon - T = b_k - \epsilon, \end{aligned}$$

which contradicts to (2.15), and (iv) is proved.

Finally, by (d) and (iii), similar to the proof of Lemma 6.8 of [21], we see that (v) also holds. \square

For τ_k -norm defined in (1.7), the same as in [16] we introduce the following definition

Definition 2.1 *Let B_k and N_k be defined in (1.8). For any $T > 0$, the mapping $\gamma : [0, T] \times B_k \rightarrow X$ is a τ_k -admissible homotopy if*

- γ is τ_k -continuous in the sense that

$$\gamma(t_m, u_m) \xrightarrow{m} \gamma(t, u) \text{ in } \tau_k\text{-topology, if } t_m \xrightarrow{m} t \text{ and } u_m \xrightarrow{m} u \text{ in } \tau_k\text{-topology.}$$

- For any $(t, u) \in [0, T] \times B_k$ there exist a neighborhood $W_{(t, u)}$ of (t, u) in the $|\cdot| \times \tau_k$ -topology such that

$$\{v - \gamma(s, v) | (s, v) \in W_{(t, u)} \cap ([0, T] \times B_k)\}$$

is contained in a finite-dimensional subspace of X .

We remark that such γ does exist since the identity mapping $I_d : I_d(t, u) \equiv u$ is a τ_k -admissible homotopy.

Let Y_k and Z_k be defined in (1.2), B_k and N_k be defined in (1.8). The following intersection lemma is proved in [16] where the author estimated the genus of $(\gamma(t, B_k) \cap N_k)$ (see also [11]).

Lemma 2.3 [16, Proposition γ] *Let $\varphi \in C^1(X, \mathbb{R})$ be an even functional and let $\gamma : [0, T] \times B_k \rightarrow X$ be a τ_k -admissible homotopy with the following properties:*

- $\gamma(0, u) = u$, for any $u \in B_k$,

- $\gamma(t, -u) = -\gamma(t, u)$,
- $\varphi(\gamma(t, u))$ is non-increasing in $t \in [0, T]$ for fixed $u \in B_k$,
- for any $t \in [0, T]$, $\gamma(t, \cdot) : B_k \rightarrow \gamma(t, B_k)$ is a τ_k -homeomorphism.

If $\sup_{u \in Y_k, \|u\|=\rho_k} \varphi(u) < \inf_{u \in Z_k, \|u\|\leq r_k} \varphi(u)$, with $0 < r_k < \rho_k$, then

$$\gamma(t, B_k) \bigcap N_k \neq \emptyset \quad \text{for any } t \in [0, T],$$

where B_k is given by (1.8).

Now we come to prove Lemma 2.1.

Proof of Lemma 2.1.

By contradiction, if the conclusion of Lemma 2.1 is false, then, there exists $\epsilon > 0$ such that

$$\|\varphi'(u)\| > \epsilon, \quad \text{for any } u \in E,$$

where E is defined by (2.2). By Lemma 2.2 we know that there exists a map $\eta(t, u) \in C([0, T] \times B_k, X)$ satisfying Lemma 2.2 (i)-(v). By Remark 1.1, the τ -topology and τ_k -topology are equivalent, then it is easy to see that $\eta(t, u)$ satisfies the assumptions of Lemma 2.3, hence,

$$\eta(T, B_k) \bigcap N_k \neq \emptyset,$$

and the definition of b_k implies that

$$\sup_{u \in B_k} \varphi(\eta(T, u)) \geq b_k.$$

However, Lemma 2.2 (iv) shows that

$$\sup_{u \in B_k} \varphi(\eta(T, u)) \leq b_k - \epsilon,$$

which leads to a contradiction. So, the proof is complete. □

Proof of Theorem 1.2.

Taking $\delta_1 > 0$, by (A₄) we know that

$$\sup_{\|u\|_\tau < \delta_1} \varphi(u) \leq C_{\delta_1},$$

for some $C_{\delta_1} > 0$. Then condition (A₃) implies that there exists $k_1 \in \mathbf{N}$ sufficiently large such that

$$b_{k_1} > \sup_{\|u\|_\tau < \delta_1} \varphi(u).$$

By Lemma 2.1, we know that there exists a sequence $\{u_n^{k_1}\}$ satisfies that

$$\varphi'(u_n^{k_1}) \xrightarrow{n} 0 \text{ in } X', \quad \sup_n \varphi(u_n^{k_1}) < d_{k_1} + 1 \text{ and } \inf_n \|u_n^{k_1}\|_\tau \geq \frac{\delta_1}{2}.$$

Since φ satisfies the $(PS)^c$ condition, $\{u_n^{k_1}\}$ has a subsequence which is convergent to a critical point u^{k_1} of φ with $\|u^{k_1}\| \geq \|u^{k_1}\|_\tau \geq \frac{\delta_1}{2}$.

Now, we take $\delta_2 > 2\|u^{k_1}\|$ and similar to the above there exists $k_2 > k_1$ large enough such that

$$b_{k_2} > \sup_{\|u\|_\tau < \delta_2} \varphi(u),$$

and we can find the second critical point u^{k_2} with $\|u^{k_2}\| \geq \frac{\delta_2}{2} > \|u^{k_1}\|$. Clearly, $u^{k_2} \neq u^{k_1}$.

Repeat the above procedures, we get a sequence of critical points $\{u^{k_m}\}$ with

$$\lim_{m \rightarrow \infty} \|u^{k_m}\| \rightarrow \infty.$$

So, the theorem is proved. \square

3 An application

The aim of this section is to apply Theorem 1.2 to prove the existence of infinitely many solutions of problem (1.9). In this section, $X = H^1(\mathbb{R}^N)$, $N \geq 3$ with the norm $\|u\|_{H^1} = (\int_{\mathbb{R}^N} (|u|^2 + |\nabla u|^2) dx)^{\frac{1}{2}}$. $L^p(a(x), \mathbb{R}^N)$ is the Lebesgue space with positive weight $a(x)$ endowed with the norm $|u|_{L_{a(x)}^p} := (\int_{\mathbb{R}^N} a(x)|u|^p dx)^{\frac{1}{p}}$, and this norm is simply denoted by $|u|_{L^p}$ if $a(x) \equiv 1$. For $r > 0$, $B(x, r) = \{x \in \mathbb{R}^N : |x| < r\}$.

The variational functional of (1.9) is defined by

$$\varphi(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx - \frac{1}{q} \int_{\mathbb{R}^N} g(x)|u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx, \quad (3.1)$$

for $u \in H^1(\mathbb{R}^N)$. By (H_1) -(H_3), $\varphi(u) \in C^1(H^1(\mathbb{R}^N))$ and

$$\varphi'(u)\phi = \int_{\mathbb{R}^N} (\nabla u \nabla \phi + V(x)u\phi) dx - \int_{\mathbb{R}^N} g(x)|u|^{q-2}u\phi dx - \int_{\mathbb{R}^N} h(x)|u|^{p-2}u\phi dx, \quad (3.2)$$

for any $\phi \in H^1(\mathbb{R}^N)$. Moreover, φ' is weakly sequentially continuous by [21, Theorem A.2].

Let $L := -\Delta + V(x)$ be the Schrödinger operator acting on $L^2(\mathbb{R}^N)$ with domain $\mathcal{D}(L) = H^2(\mathbb{R}^N)$. Since L is self-adjoint and 0 lies in a gap of the spectrum of L , by the standard spectral theory we know that the space $H^1(\mathbb{R}^N)$ can be decomposed as $H^1(\mathbb{R}^N) = Y \oplus Z$ such that the quadratic form:

$$u \in H^1(\mathbb{R}^N) \rightarrow \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \quad (3.3)$$

is negative and positive definite on Y and Z respectively, and both Y and Z may be infinite-dimensional. Let $X = \mathcal{D}(|L|^{\frac{1}{2}})$ be equipped with the inner product

$$\langle u, v \rangle_1 := \langle |L|^{\frac{1}{2}} u, |L|^{\frac{1}{2}} v \rangle_{L^2}, \quad (3.4)$$

and norm

$$\|u\| := \langle |L|^{\frac{1}{2}}u, |L|^{\frac{1}{2}}u \rangle_{L^2}^{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle_{L^2}$ is the usual inner product in $L^2(\mathbb{R}^N)$. By the condition (H_1) , similar to the appendix of [8], we know that $X = H^1(\mathbb{R}^N)$ and the norms $\|\cdot\|$ and $\|\cdot\|_{H^1}$ are equivalent. Moreover, Y and Z are also orthogonal with respect to $\langle \cdot, \cdot \rangle_1$.

Let $P : X \rightarrow Y$ and $Q : X \rightarrow Z$ be the orthogonal projections, (3.1) can be written as

$$\varphi(u) := \frac{1}{2}(-\|Pu\|^2 + \|Qu\|^2) - \frac{1}{q} \int_{\mathbb{R}^N} g(x)|u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx. \quad (3.5)$$

Now, we set

$$Y_k := Y \oplus (\oplus_{j=0}^k \mathbb{R}f_j) \text{ and } Z_k := \overline{\oplus_{j=k}^{\infty} \mathbb{R}f_j},$$

where $\{f_j\}_{j \geq 0}$ is an orthonormal basis of $(Z, \|\cdot\|)$.

Before proving Theorem 1.3, we give some useful lemmas. The first lemma is the following embedding result which has been used in many papers (see, e.g., [18]). Here we give a simple proof for completeness.

Lemma 3.1 *If $1 < q < 2^*$ and $a(x) \in L^{q_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $a(x) \geq 0$ a.e. in \mathbb{R}^N , where $q_0 = \frac{2N}{2N-qN+2q}$. Then, $H^1(\mathbb{R}^N) \hookrightarrow L^q(a(x), \mathbb{R}^N)$ is compact.*

Proof. For $u \in H^1(\mathbb{R}^N)$, by the Hölder and Sobolev inequalities, we see that

$$\begin{aligned} \int_{\mathbb{R}^N} a(x)|u|^q dx &\leq |a(x)|_{L^{q_0}} \cdot \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} dx \right)^{\frac{qN-2q}{2N}} \\ &= |a(x)|_{L^{q_0}} \cdot |u|_{L^{2^*}(\mathbb{R}^N)}^q \leq C\|u\|^q, \end{aligned}$$

that is, $|u|_{L^q_{a(x)}} \leq C\|u\|$, which means that $H^1(\mathbb{R}^N) \hookrightarrow L^q(a(x), \mathbb{R}^N)$. Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$, passing to a subsequence, we may assume that, for some $u \in H^1(\mathbb{R}^N)$,

$$u_n \xrightarrow{n} u \text{ weakly in } H^1(\mathbb{R}^N) \text{ and } u_n \xrightarrow{n} u \text{ in } L^q_{loc}(\mathbb{R}^N), \text{ for } 1 < q < 2^*.$$

To show that $H^1(\mathbb{R}^N) \hookrightarrow L^q(a(x), \mathbb{R}^N)$ is compact, we need only to show that u_n strongly converges to u in $L^q(a(x), \mathbb{R}^N)$ for $q \in (1, 2^*)$. In fact, for any $R > 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} a(x)|u_n - u|^q dx &= \int_{\mathbb{R}^N \setminus B(0, R)} a(x)|u_n - u|^q dx + \int_{B(0, R)} a(x)|u_n - u|^q dx \\ &\leq \left(\int_{\mathbb{R}^N \setminus B(0, R)} |a(x)|^{q_0} dx \right)^{\frac{1}{q_0}} |u_n - u|_{L^{2^*}(\mathbb{R}^N)} + |a|_{L^\infty(\mathbb{R}^N)} \int_{B(0, R)} |u_n - u|^q dx \\ &\rightarrow 0, \end{aligned}$$

by letting $n \rightarrow +\infty$ and then $R \rightarrow +\infty$. □

Lemma 3.2 *Under condition (H_3) , let*

$$\beta_k := \sup_{u \in Z_k, \|u\|=1} |u|_{L_{h(x)}^p}, \text{ for any } k \in \mathbf{N},$$

then, $\beta_k \rightarrow 0$ as $k \rightarrow \infty$.

Proof. It is clear that $0 < \beta_{k+1} \leq \beta_k$, and then $\beta_k \rightarrow \beta \geq 0$ as $k \rightarrow \infty$. For every $k \geq 0$, there exists $u_k \in Z_k$ with $\|u_k\| = 1$ and $|u_k|_{L_{h(x)}^p} \geq \frac{\beta_k}{2}$. By the definition of Z_k , we have $u_k \xrightarrow{k} 0$ in $H^1(\mathbb{R}^N)$. Thus Lemma 3.1 implies that $u_k \xrightarrow{n} 0$ strongly in $L_{g(x)}^q$, then, $\beta = 0$. \square

Lemma 3.3 *If $(H_1) - (H_3)$ hold, then φ satisfies $(PS)^c$ condition in $H^1(\mathbb{R}^N)$ for any $c < +\infty$.*

Proof. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be any sequence satisfying

$$\sup_n \varphi(u_n) \leq c \text{ and } \varphi'(u_n) \xrightarrow{n} 0.$$

We claim that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Indeed, for n large, there holds

$$\begin{aligned} c + 1 + \|u_n\| &\geq \varphi(u_n) - \frac{1}{2} \langle \nabla \varphi(u_n), u_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} h(x) |u_n|^p dx + \left(\frac{1}{2} - \frac{1}{q}\right) \int_{\mathbb{R}^N} g(x) |u_n|^q dx, \end{aligned}$$

that is,

$$\begin{aligned} \int_{\mathbb{R}^N} h(x) |u_n|^p dx &\leq C + \|u_n\| + C \int_{\mathbb{R}^N} |g(x)| |u_n|^q dx. \\ &\leq C + \|u_n\| + C \|u_n\|^q. \end{aligned} \tag{3.6}$$

Let $u_n = y_n + z_n$, with $y_n \in Y, z_n \in Z$. For n large,

$$\|z_n\| \geq \langle \varphi'(u_n), z_n \rangle = \|z_n\|^2 - \int_{\mathbb{R}^N} g(x) |u|^{q-2} u z_n dx - \int_{\mathbb{R}^N} h(x) |u|^{p-2} u z_n dx,$$

thus

$$\|z_n\|^2 \leq \|z_n\| + \int_{\mathbb{R}^N} g(x) |u|^{q-1} z_n dx + \int_{\mathbb{R}^N} h(x) |u|^{p-1} z_n dx.$$

By Hölder inequality and Sobolev embeddings, we see that, for some $C > 0$,

$$\begin{aligned} \|z_n\|^2 &\leq \|z_n\| + |g(x)^{\frac{q-1}{q}} u_n^{q-1}|_{L^{\frac{q}{q-1}}} |g(x)^{\frac{1}{q}} z_n^{q-1}|_{L^{\frac{q}{q-1}}} + |h(x)^{\frac{p-1}{p}} u_n^{p-1}|_{L^{\frac{p}{p-1}}} |h(x)^{\frac{1}{p}} z_n^{p-1}|_{L^{\frac{p}{p-1}}} \\ &= \|z_n\| + |u_n|_{L_{g(x)}^q}^{q-1} |z_n|_{L_{g(x)}^q} + \left(\int_{\mathbb{R}^N} h(x) |u|^p dx \right)^{\frac{p-1}{p}} |z_n|_{L_{h(x)}^p} \\ &\leq \|z_n\| + C \|u_n\|^{q-1} \|z_n\| + C(1 + \|u_n\| + \|u_n\|^q)^{\frac{p-1}{p}} \|z_n\|, \text{ by (3.6),} \end{aligned}$$

$$\leq \|u_n\| + C\|u_n\|^q + C(1 + \|u_n\| + \|u_n\|^q)^{\frac{p-1}{p}}\|u_n\|.$$

Similarly, it follows from $\|y_n\| \geq -\langle \varphi'(u_n), y_n \rangle$ that

$$\begin{aligned} \|y_n\|^2 &\leq \|y_n\| + |g(x)^{\frac{q-1}{q}} u_n^{q-1}|_{L^{\frac{q}{q-1}}} |g(x)^{\frac{1}{q}} y_n^{q-1}|_{L^{\frac{q}{q-1}}} + |h(x)^{\frac{p-1}{p}} u_n^{p-1}|_{L^{\frac{p}{p-1}}} |h(x)^{\frac{1}{p}} y_n^{p-1}|_{L^{\frac{p}{p-1}}} \\ &\leq \|u_n\| + C\|u_n\|^q + C(1 + \|u_n\| + \|u_n\|^q)^{\frac{p-1}{p}}\|u_n\|. \end{aligned}$$

Since $\|u_n\|^2 = \|y_n\|^2 + \|z_n\|^2$, the above conclusions show that

$$\|u_n\|^2 \leq 2\|u_n\| + C\|u_n\|^q + C(1 + \|u_n\| + \|u_n\|^q)^{\frac{p-1}{p}}\|u_n\|.$$

Thus, by $q < \frac{p}{p-1}$, we know $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

By the boundedness of $\{u_n\}$, we may assume, up to a subsequence, that

$$y_n \xrightarrow{n} y \text{ in } H^1(\mathbb{R}^N) \text{ and } z_n \xrightarrow{n} z \text{ in } H^1(\mathbb{R}^N).$$

Let $u = y + z$, we get that

$$\langle \nabla \varphi(u_n) - \nabla \varphi(u), y_n - y \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$\begin{aligned} \langle \varphi'(u_n) - \varphi'(u), y_n - y \rangle &= -\|y_n - y\|^2 + \int_{\mathbb{R}^N} g(x)(|u|^{q-2}u - |u_n|^{q-2}u_n)(y_n - y)dx \\ &\quad + \int_{\mathbb{R}^N} h(x)(|u|^{p-2}u - |u_n|^{p-2}u_n)(y_n - y)dx. \end{aligned}$$

Using the Hölder inequality, we see that

$$y_n \xrightarrow{n} y \text{ in } H^1(\mathbb{R}^N).$$

Similarly,

$$z_n \xrightarrow{n} z \text{ in } H^1(\mathbb{R}^N).$$

Hence,

$$u_n \xrightarrow{n} u \text{ in } H^1(\mathbb{R}^N).$$

So, we proved that φ satisfies the $(PS)^c$ condition for any $c < +\infty$.

□

Lemma 3.4 *Under the conditions $(H_1) - (H_3)$, for any $\delta > 0$, there exists $C_\delta < \infty$ such that $\sup_{\|u\|_\tau < \delta} \varphi(u) < C_\delta$.*

Proof. By $u \in H^1(\mathbb{R}^N) = Y \oplus Z$ with $Z = Y^\perp$ we may set $u = y + z$ for some $y \in Y$ and $z \in Z$, then,

$$\begin{aligned}\varphi(u) &= \frac{1}{2}(-\|y\|^2 + \|z\|^2) - \frac{1}{q} \int_{\mathbb{R}^N} g(x)|u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx \\ &\leq \frac{1}{2}(-\|y\|^2 + \|z\|^2) + \frac{1}{q} \int_{\mathbb{R}^N} |g(x)||u|^q dx \\ &\leq -\frac{1}{2}\|y\|^2 + C\|y\|^q + \frac{1}{2}\|z\|^2 + C\|z\|^q.\end{aligned}$$

Since $q < 2$, $-\frac{1}{2}\|y\|^2 + C\|y\|^q$ is bounded from above. By (1.4), we have $\|z\| \leq \|u\|_\tau \leq \delta$, so, there exists $C_\delta < \infty$ such that

$$\sup_{\|u\|_\tau < \delta} \varphi(u) < C_\delta.$$

□

Proof of Theorem 1.3. By Theorem 1.2 and Lemmas 3.3-3.4, in order to prove Theorem 1.3 we need only to verify the conditions (A_1) , $(A_2)'$ and (A_3) .

Clearly, (A_1) is true since φ maps a bounded set into a bounded set.

Next, we prove $(A_2)'$ by showing that $a_k \rightarrow -\infty$ as $\rho_k \rightarrow \infty$. Let $u = y + z$ with $y \in Y$ and $z \in Z$. For any $u \in Y_k$, by Lemma 3.1, we see that

$$\begin{aligned}\varphi(u) &= \frac{1}{2}(-\|y\|^2 + \|z\|^2) - \frac{1}{q} \int_{\mathbb{R}^N} g(x)|u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx \\ &\leq \frac{1}{2}(-\|y\|^2 + \|z\|^2) + \frac{1}{q} |u|_{L_{|g(x)|}^q}^q - \frac{1}{p} |u|_{L_{h(x)}^p}^p \\ &\leq \frac{1}{2}(-\|y\|^2 + \|z\|^2) + \frac{2^{q-1}}{q} (\|y\|_{L_{|g(x)|}^q}^q + \|z\|_{L_{|g(x)|}^q}^q) - \frac{1}{p} |u|_{L_{h(x)}^p}^p \\ &\leq \frac{1}{2}(-\|y\|^2 + \|z\|^2) + C_q(\|y\|^q + \|z\|^q) - \frac{1}{p} |u|_{L_{h(x)}^p}^p.\end{aligned}$$

Since $H^1(\mathbb{R}^N) \hookrightarrow L^p(h(x), \mathbb{R}^N)$, we denote by E_k the closure of Y_k in $L^p(h(x), \mathbb{R}^N)$, then there exists a continuous projection from E_k to $\bigoplus_{j=0}^k f_j$, thus there exists a constant $C > 0$ such that

$$|z|_{L_{h(x)}^p}^p \leq C |u|_{L_{h(x)}^p}^p,$$

and note that all norms are equivalent in a finite-dimensional vector space, then for any $z \in \bigoplus_{j=0}^k f_j$, there exists $C > 0$ such that

$$\|z\|^p \leq C |z|_{L_{h(x)}^p}^p,$$

thus

$$\varphi(u) \leq (-\frac{1}{2}\|y\|^2 + C\|y\|^q) + (\frac{1}{2}\|z\|^2 + C\|z\|^q - C\|z\|^p).$$

So,

$$a_k := \sup_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \rightarrow -\infty \quad \text{as } \rho_k \rightarrow \infty.$$

Finally, for any $u \in Z_k$ with $\|u\| = r_k$, let $u = y + z$ with $y \in Y$ and $z \in Z$, then $y = 0$, $z = u$. Furthermore,

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{q} \int_{\mathbb{R}^N} g(x)|u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{q} \int_{\mathbb{R}^N} |g(x)||u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx \\ &\geq \left(\frac{1}{4}\|u\|^2 - C\|u\|^q\right) + \left(\frac{1}{4}\|u\|^2 - \frac{1}{p}\beta_k^p\|u\|^p\right).\end{aligned}$$

Choose $r_k = \left(\frac{p}{4}\right)^{\frac{1}{p-2}} \frac{1}{\beta_k^{\frac{p}{p-2}}}$, we have

$$\varphi(u) \geq \frac{1}{4}\|u\|^2 - C\|u\|^q = \frac{1}{4}|r_k|^2 - C|r_k|^q.$$

Since we have $r_k \xrightarrow{k} +\infty$ by

$$\beta_k \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

then,

$$b_k := \inf_{u \in Z_k, \|u\|=r_k} \varphi(u) \rightarrow +\infty \quad \text{as } k \rightarrow \infty.$$

Thus (A_3) is also proved.

Acknowledgements: This work was supported by the NSFC under grants 11471331 and 11501555.

□

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